Triggered by EWD1183

In EWD1183 — "Less than" in terms of "at most" — Edsger W. Dijkstra presents three definitions of $<$ in terms of $\leq$. The first one applies if $\leq$ is a total order and it reads

$$D0: \quad x < y \equiv \neg(y \leq x)$$

The second one applies if $\leq$ not necessarily is a total order; it reads

$$D1: \quad x < y \equiv x \leq y \wedge x \neq y$$

And then Dijkstra writes: "In [GS94] — "A logical approach to discrete math" by Gries & Schneider — I found the much nicer definition:". And then he mentions

$$GS: \quad x < y \equiv x \leq y \wedge \neg(y \leq x)$$

The sentence just quoted sounds as if definition GS is an alternative to D1. But it is not; the purpose of this note is to find out under what circumstances the three definitions of $<$ are mutually equivalent.

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$\langle \forall x,y :: D0 \equiv D1 \rangle$?

This is the case, precisely when the right hand sides of D0 and D1 are equivalent, i.e. when

$$\neg(y \leq x) \equiv x \leq y \wedge x \neq y \qquad (\forall x,y)$$

We examine the $\Rightarrow$ - and $\Leftarrow$ - part of this expression separately.

$$\begin{array}{ll}
\bullet & \neg(y \leq x) \Rightarrow x \leq y \wedge x \neq y \\
\equiv & \quad \{ (P\Rightarrow) \text{ over } \wedge \} \\
 & (\neg(y \leq x) \Rightarrow x \leq y) \wedge (\neg(y \leq x) \Rightarrow x \neq y) \\
\equiv & \quad \{ \text{pred. calc.} \} \\
 & (y \leq x \vee x \leq y) \wedge (x = y \Rightarrow y \leq x) \\
\equiv & \quad \{ \forall x,y \} \\
 & (\leq \text{ is total}) \wedge (\leq \text{ is reflexive})
\end{array}$$

$$\begin{array}{ll}
\bullet & x \leq y \wedge x \neq y \Rightarrow \neg(y \leq x) \\
\equiv & \quad \{ \text{shunting, twice} \} \\
 & x \leq y \wedge y \leq x \Rightarrow x = y \\
\equiv & \quad \{ \forall x,y \} \\
 & \leq \text{ is antisymmetric} .
\end{array}$$

So the answer is that D0 and D1 are equivalent just when

$\leq$ is reflexive, antisymmetric, and total .

$\langle \forall x,y :: D0 \equiv GS \rangle$?

We again equate the two right hand sides and deal with the two implications separately.

- $\neg(y \le x) \Rightarrow x \le y \wedge \neg(y \le x)$

$\equiv$ { absorption }

$\neg(y \le x) \Rightarrow x \le y$

$\equiv$ { pred. calc. }

$y \le x \vee x \le y$

$\equiv$ { $\forall x, y$ }

$\le$ is total

- $x \le y \wedge \neg(y \le x) \Rightarrow \neg(y \le x)$

$\equiv$ { pred. calc. }

true .

So, D0 and GS are equivalent just when $\le$ is total .

And from this, we can already see that D1 and GS are not equivalent (Can someone guess when they are? I couldn't.)

$\langle \forall x, y :: D1 \equiv GS \rangle$?

As before,

- $x \le y \wedge x \ne y \Rightarrow x \le y \wedge \neg(y \le x)$

$\equiv$ { pred. calc. }

$x \le y \wedge x \ne y \Rightarrow \neg(y \le x)$

$\equiv$ { shunting, twice }

$x \le y \wedge y \le x \Rightarrow x = y$

$\equiv$ { $\forall x, y$ }

$\le$ is antisymmetric

$$
\begin{array}{ll}
\bullet & x \le y \wedge \neg(y \le x) \;\Rightarrow\; x \le y \wedge x \ne y \\
\equiv & \quad \{\text{pred. calc.}\} \\
 & x \le y \wedge \neg(y \le x) \;\Rightarrow\; x \ne y \\
\equiv & \quad \{\text{shunting, twice}\} \\
 & x \le y \wedge x = y \;\Rightarrow\; y \le x \\
\equiv & \quad \{\text{Leibniz, twice}\} \\
 & y \le x \wedge x = y \;\Rightarrow\; y \le x \\
\equiv & \quad \{\text{pred. calc.}\} \\
 & \text{true}
\end{array}
$$

Hence, D1 and GS are equivalent just when

$\leq$ is antisymmetric .

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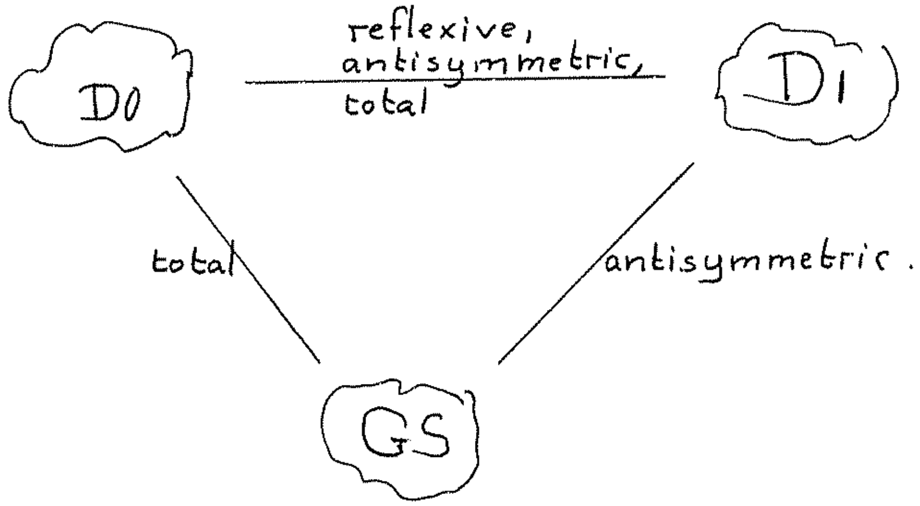
We can summarize the above by



What happened to the reflexivity along the path D0 - GS - D1 ? It was only when I saw the above that it dawned upon me that totality implies reflexivity.

12 January 1995
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