A very clever and impressive (but highly intractable) proof of a beautiful theorem.

With r and s relations, the beautiful theorem reads

$$0 \wedge 1 \wedge 2 \Rightarrow 3 \quad ,$$

where

0: r is well-founded
1: s is well-founded
2: $r \vee s$ is transitive
3: $r \vee s$ is well-founded

* * *

It was

Henk Doornbos

who communicated this theorem. Quite a number of people have tried to prove it within the relational calculus, and they all failed; except for Henk Doornbos who after a long, long while and after many serious efforts encountered a proof. Henk incorporated his proof in his forthcoming PhD-thesis and provided a heuristics. But that heuristics is so unconvincing that we leave it out here. We just present Henk's proof as one long calculation. The main characteristic of the proof is that the right thing is done at the right moment, and that nobody knows when what is right. The reason for extracting this passage from Henk's PhD-thesis, is the problem's right of existence all by itself.

* * *

We formalize 0, 1, 2, and 3, thereby using Rutger M. Dijkstra's formulation of well-foundedness.

(0) $\langle \forall x :: [x \Rightarrow x;r] \Rightarrow [x \Rightarrow false] \rangle$

(1) $\langle \forall x :: [x \Rightarrow x;s] \Rightarrow [x \Rightarrow false] \rangle$

(2) $[(r \vee s);(r \vee s) \Rightarrow r \vee s]$

(3) $\langle \forall x :: [x \equiv x;(r \vee s)] \Rightarrow [x \Rightarrow false] \rangle$

Explanation Relation r is well-founded means that equation $x: [x \equiv x;r]$ has $false$ as its weakest solution. By Knaster-Tarski, (0) is the "extremity part" of r's well-foundedness. There is little point in writing down the "solves-part" in this case.
End of Explanation.

We now prove (3) that, for any x, (3)'s consequent is true. We thereby use (0), (1), (2), and

(4) $[x \equiv x;(r \vee s)]$,

the antecedent of (3).
Off we go.

$[x \Rightarrow false]$

$\Leftarrow$ { (0) }

$[x \Rightarrow x;r]$

$\equiv$ { shunting }

$[x \wedge \neg(x;r) \Rightarrow false]$

$\equiv$ { • let y be such that $[y \equiv x \wedge \neg(x;r)]$ }

$[y \Rightarrow false]$

$\Leftarrow$ { (1) }

$[y \Rightarrow y;s]$

$\equiv$ { Lemma 0, to be shown later:
$[y \equiv x;s \wedge \neg(x;r)]$
(proof uses (4)) }

$[x;s \wedge \neg(x;r) \Rightarrow y;s]$

$\equiv$ { shunting }

$[x;s \Rightarrow x;r \vee y;s]$

$\equiv$ { by (4), $[x \equiv x;(r \vee s)^*]$ }

$[x;s \Rightarrow x;(r \vee s)^*;r \vee y;s]$

$\Leftarrow$ { $[x \Leftarrow y]$, from definition of y }

$[x;s \Rightarrow y;(r \vee s)^*;r \vee y;s]$

$\Leftarrow$ { • let u be such that
$[u;s \Rightarrow (r \vee s)^*;r \vee s]$ }

$[x;s \Rightarrow y;u;s]$

$\Leftarrow$ { $(;s)$ is monotonic }

$[x \Rightarrow y;u]$

$\equiv$ { shunting }

$[x \wedge \neg(y;u) \Rightarrow \text{false}]$

$\Leftarrow$ { (0) with $x := x \wedge \neg(y;u)$ }

$[x \wedge \neg(y;u) \Rightarrow (x \wedge \neg(y;u));r]$

$\Leftarrow$ { Lemma 1, to be shown below:
$[(a \wedge \neg b);c \Leftarrow a;c \wedge \neg(b;c)]$ }

$[x \wedge \neg(y;u) \Rightarrow x;r \wedge \neg(y;u;r)]$

$\equiv$ { shunting }

$[x \Rightarrow y;u \vee (x;r \wedge \neg(y;u;r))]$

$\equiv$ { predicate calculus }

$[x \Rightarrow y;u \vee x;r]$
$\wedge [x \Rightarrow y;u \vee \neg(y;u;r)]$

$\Leftarrow$ { definition of y & shunting }

$[y \Rightarrow y;u] \wedge [x \wedge y;u;r \Rightarrow y;u]$

$\Leftarrow$ { relational and predicate calculus }

$[J \Rightarrow u] \wedge [u;r \Rightarrow u]$

$\equiv$ { predicate calculus }

$[J \vee u;r \Rightarrow u]$

$\Leftarrow$ { definition of $*$ }

$[u \equiv r^*]$

$\equiv$ { • let u be such that $[u \equiv r^*]$ }

true .

And this completes the "main calculation"

The remaining obligations are to check the solvability of the requirements on u, and to prove Lemma 0 and Lemma 1.

Proof of Lemma 0

$$
\begin{array}{ll}
 & y \\
\equiv & \quad \{\text{definition of } y\} \\
 & x \wedge \neg(x;r) \\
\equiv & \quad \{(4)\} \\
 & x;(r \vee s) \wedge \neg(x;r) \\
\equiv & \quad \{; \text{ over } \vee\} \\
 & (x;r \vee x;s) \wedge \neg(x;r) \\
\equiv & \quad \{\text{predicate calculus}\} \\
 & x;s \wedge \neg(x;r)
\end{array}
$$

End

Proof of Lemma 1

For any a, b, c we have

$$
\begin{array}{ll}
 & [(a \wedge \neg b);c \Leftarrow a;c \wedge \neg(b;c)] \\
\equiv & \quad \{\text{shunting}\} \\
 & [(a \wedge \neg b);c \vee b;c \Leftarrow a;c] \\
\Leftarrow & \quad \{\text{relational calculus}\} \\
 & [(a \wedge \neg b) \vee b \Leftarrow a] \\
\equiv & \quad \{\text{predicate calculus}\} \\
 & \text{true} .
\end{array}
$$

End

Re u We have to show $[r^*;s \Rightarrow (r\vee s)^*;r \vee s]$
This is the place where the transitivity of $r\vee s$, i.e. (2), will enter the game. And the only thing we use of it is

(5) $[r;s \Rightarrow r\vee s]$

We observe

$$
\begin{array}{ll}
 & [r^*;s \Rightarrow (r\vee s)^*;r \vee s] \\
\equiv & \{\bullet \text{ let } z \text{ be such that } [z \equiv (r\vee s)^*;r \vee s]\ \} \\
 & [r^*;s \Rightarrow z] \\
\Leftarrow & \{\ r^*;s \text{ is the strongest solution of } x: [s \vee r;x \Rightarrow x]\ \} \\
 & [s \vee r;z \Rightarrow z] \\
\equiv & \{\ [s \Rightarrow z]\ \} \\
 & [r;z \Rightarrow z] \\
\equiv & \{\ \text{definition of } z\ \} \\
 & [r;(r\vee s)^*;r \Rightarrow z] \wedge [r;s \Rightarrow z] \\
\Leftarrow & \{\ (5)\ \} \\
 & [r;(r\vee s)^*;r \Rightarrow z] \wedge [r\vee s \Rightarrow z] \\
\equiv & \{\ \text{right conjunct} \equiv \mathit{true}\ \} \\
 & [r;(r\vee s)^*;r \Rightarrow z] \\
\Leftarrow & \{\ [r \Rightarrow r\vee s]\ \} \\
 & [(r\vee s);(r\vee s)^*;r \Rightarrow z] \\
\equiv & \{\ \text{definition of } z\ \} \\
 & \mathit{true}\ .
\end{array}
$$

End

* * *

Of course, the question is: Can we do better?

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