A decomposition theorem for multibounds

In this note we record a simple theorem that came up in the construction of a progress argument for a multiprogram. It is, in fact, a theorem about maxima and minima.

Exhibiting a so-called multibound for a set Z of integers (integer variables) means exhibiting a natural number m such that

(0) $\uparrow Z \le \downarrow Z + m$ (*) .

We show that in order to do so for set Z, where

(1) $Z = X \cup Y$,

it suffices to exhibit two weaker (see Ping below) multibounds, viz. (2) and (3) given by

(2) $\uparrow X \le \downarrow Y + e$

(3) $\uparrow Y \le \downarrow X + f$.

More precisely, we prove

<u>Theorem</u> For $Z = X \cup Y$,

$$\langle \exists m :: (0) \rangle \equiv \langle \exists e :: (2) \rangle \wedge \langle \exists f :: (3) \rangle .$$

<u>Proof</u> The proof is a ping-pong argument. This seems to be unavoidable, given the shape of the demonstrandum.

(*) : $\uparrow$ and $\downarrow$ denote the maximum and minimum, respectively.

<u>Ping</u>. We deal with (2); then (3) will follow on account of symmetry. Just using (1) -twice- and (0) we have:

$$\uparrow X \leq \uparrow Z \leq \downarrow Z + m \leq \downarrow Y + m .$$

<u>Pong</u>. In fact, all the work of the proof is in this part. The properties that we shall use more or less implicitly are

- $\uparrow Z = (\uparrow X) \uparrow (\uparrow Y) \quad \wedge \quad \downarrow Z = (\downarrow X) \downarrow (\downarrow Y)$
- $a \uparrow b \leq c \quad \equiv \quad a \leq c \quad \wedge \quad b \leq c$
- $c \leq a \downarrow b \quad \equiv \quad c \leq a \quad \wedge \quad c \leq b$.
- addition distributes over $\uparrow$ and $\downarrow$.

We shall deal with (0) by calculating that

$\uparrow X \leq \downarrow Z + m$ for $e + f \leq m$.

Then $\uparrow Y \leq \downarrow Z + m$ for $e + f \leq m$, by symmetry.

And then we are done. We calculate:

$$\begin{array}{llcll}
 & \uparrow X & \text{and} & & \uparrow X \\
\leq & \{(2)\} & & \leq & \{(2)\} \\
 & \downarrow Y + e & & & \downarrow Y + e \\
\leq & \{\downarrow \uparrow\} & & \leq & \{\text{widening}, 0 \leq f\} \\
 & \uparrow Y + e & & & \downarrow Y + e + f \\
\leq & \{(3)\} & & & \\
 & \downarrow X + f + e & & &
\end{array}$$

So $\uparrow X \leq (\downarrow X) \downarrow (\downarrow Y) + e + f$.

■

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