Exercises in Calculating

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Dear Masters, Colleagues, and Students at the 1996 Marktoberdorf Summer School,

When Edsger W. Dijkstra indicated that he did not feel up to attending this year's Summer School, one of us (WF) was invited to fill in his slots. Prof. Dijkstra had planned to address "The Design of Calculations", an important issue for modern computing science. Due to shortage of time we were not able to compose a coherent and informative text dealing with this topic. So we decided to fill in the empty slots with some talks on Multiprogramming.

Yet, we felt that, in one way or another, we had to remain faithful to Dijkstra's original proposal, and that is why we decided to concoct a small set of exercises that can serve as a carrier for discussing all sorts of aspects concerning the art of calculating. Our plan is to distribute the exercises few by few —i.e. not all at a time—, and ask you to solve and discuss them during dinner or lunch, or in a lost hour, or even late at night in a pub. We can ask this because the bulk of our exercises do not require more than, say, 10 calculational steps, and most of them even far less than that.

There is one important thing, though, that you must promise us not to do, namely regard them as exercises in problem solving. Of course you are welcome to feel excited and delighted when you found/designed a calculational solution, but you are not allowed to leave it at that, because it is only then when the real game starts. Because it is only then that you, in discussion with others who tackled the problem, should address questions like

- which have been my design considerations?
- what is the overall structure of my calculation, and could it have been done differently?
- what is the quality of the hints justifying the correctness of the individual steps? (An important quality criterion is that a reader can read (and understand) your calculation at a steady, not too slow pace.)
- are the individual steps too small, too naive, or, on the contrary, too large?
- how is the layout of my calculation on paper? Is the spacing attractive to the human mind? (For instance: "$x^2+y^2=(x+y)^2 \equiv \{\text{expansion}\}\ x^2+y^2=x^2+2xy+y^2 \equiv \{\text{cancellation}\}\ 0=2xy \equiv \{\text{algebra}\}\ 0=x \vee 0=y$" is not!)
- etcetera

Of course, the people present at this Summer School have lots of different backgrounds. They have different mother tongues, different academic educations, and different ages. And —there is no escaping it— these differences will pop up in the discussions. In dealing with these exercises, however, we must agree on something, and this is

- that we are supposed to be more or less familiar with elementary predicate calculus, for instance as described in [DS90]
- that we stick to the kind of calculational format as used in, for instance, [Gas90], [DS90], and [GS93].

Among the people present at this Summer School, there are a number who are familiar with the intended calculational style, such as David Gries, Rutger M. Dijkstra, Wim Feijen, Netty van Gasteren, Markus Kaltenbach, Burghard von Karger, and definitely some others. Please, don't hesitate to contact them/us.

We conclude this little letter with the warning that some of the exercises are extremely simple, and that some are quite hard. Good luck, and ... enjoy!

References

[DS90] E.W. Dijkstra and C.S. Scholten,
Predicate Calculus and
Program Semantics,
Springer-Verlag, Berlin, 1990

[Gas90] A.J.M. van Gasteren,
On the Shape of
Mathematical Arguments,
Lecture Notes in Computer Science 445,
Springer-Verlag, New York, 1990

[GS93] D. Gries and F.B. Schneider,
A Logical Approach to Discrete Math,
Springer-Verlag, New York, 1993

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Exercise 0

We consider an anonymous universe equipped with a relation $\leq$.

- For reflexive and antisymmetric $\leq$, we have the two so-called "Rules of Indirect Equality", viz. for all x, y:

$$x = y \quad \equiv \quad \langle \forall z :: \; z \leq x \equiv z \leq y \rangle \quad ,$$

$$x = y \quad \equiv \quad \langle \forall z :: \; x \leq z \equiv y \leq z \rangle \quad .$$

Prove one of them. (Hint: give a ping-pong proof, i.e. a proof by mutual implication.)

- For reflexive and transitive $\leq$, we have the two so-called "Rules of Indirect Inequality", viz. for all x, y:

$$x \leq y \quad \equiv \quad \langle \forall z :: \; z \leq y \Leftarrow z \leq x \rangle \quad ,$$

$$x \leq y \quad \equiv \quad \langle \forall z :: \; x \leq z \Leftarrow y \leq z \rangle \quad .$$

Prove one of them.

Exercise 1 (from the very beginning of lattice-calculus)

We consider an anonymous universe equipped with a reflexive, antisymmetric relation $\leq$ and a binary infix operator $\uparrow$ ("up"), related by, for all x, y, z :

$$x \uparrow y \leq z \quad \equiv \quad x \leq z \ \wedge \ y \leq z \quad .$$

Prove

a) $y \leq x \uparrow y$

b) $\uparrow$ is associative

c) $x \uparrow y = y \quad \equiv \quad x \leq y$

d) $\leq$ is transitive

e) For each function f to and from our universe,

f distributes over $\uparrow$
$\Rightarrow$ f is monotonic with respect to $\leq$.

f) $z \leq x \uparrow y \quad \Leftarrow \quad z \leq x \ \vee \ z \leq y$

g) For $\leq$ total as well, i.e. $p \leq q \ \vee \ q \leq p$ for all p, q :

$$z \leq x \uparrow y \quad \equiv \quad z \leq x \ \vee \ z \leq y \quad .$$

Exercise 2 (from the beginning of lattice-calculus)

We consider a universe with reflexive, antisymmetric, and transitive relation $\leq$. Furthermore, for each predicate R and endo-function t, i.e. a function to and from the universe,

$$\langle \uparrow x : R.x : t.x \rangle$$

is given to be an element of the universe; by definition it satisfies

$$\langle \uparrow x : R.x : t.x \rangle \leq z \quad \equiv \quad \langle \forall x : R.x : t.x \leq z \rangle \ .$$

Prove

a) [Instantiation]

$$R.x \quad \Rightarrow \quad t.x \leq \langle \uparrow y : R.y : t.y \rangle$$

b) [Term monotonicity]

$$\langle \forall x :: s.x \leq t.x \rangle$$

$$\Rightarrow$$

$$\langle \uparrow x :: s.x \rangle \leq \langle \uparrow x :: t.x \rangle \quad ,$$ for each range

c) [Range Monotonicity]

$$\langle \forall x :: R.x \Rightarrow S.x \rangle$$

$$\Rightarrow$$

$$\langle \uparrow x : R.x : t.x \rangle \leq \langle \uparrow x : S.x : t.x \rangle$$

d) f is monotonic with respect to $\leq$

$\equiv$

for each x, $f.x = \langle \uparrow y : y \leq x : f.y \rangle$

Similar, dual properties hold for

$\langle \downarrow x : R.x : t.x \rangle$,

defined by

$$z \leq \langle \downarrow x : R.x : t.x \rangle \quad \equiv \quad \langle \forall x : R.x : z \leq t.x \rangle \quad .$$

(Operator $\uparrow$ is commonly called "supremum" and $\downarrow$ "infimum".)

Exercise 3 (from the theory of Galois Connections)

We consider a universe with reflexive, antisymmetric, and transitive relation $\leq$. Furthermore, for each predicate R and endofunction t, supremum $\langle \uparrow x : R.x : t.x \rangle$ and infimum $\langle \downarrow x : R.x : t.x \rangle$ are defined. (See Exercise 2.)

Let f and g be two endofunctions coupled by the so-called Galois Connection : for all x, y

$$(*) \qquad f.x \leq y \quad \equiv \quad x \leq g.y \quad .$$

Prove

a) f distributes over arbitrary suprema. (This is commonly called : "f is universally $\uparrow$-junctive".)

b) g distributes over arbitrary infima.

c) [Rules of Cancellation]

$$x \leq g.(f.x)$$
$$f.(g.x) \leq x \quad .$$

d) $$f.x = \langle \downarrow z : x \leq g.z : z \rangle$$
$$g.y = \langle \uparrow z : f.z \leq y : z \rangle$$

e) Monotonic functions f and g that satisfy the Rules of Cancellation are Galois-Connected as in (*) above.

f) For universally ↑-junctive f, there exists a function g satisfying.
(In the jargon, g is called an "upper-adjoint" of f.)

g) Upper (and lower) adjoints are unique.

Exercise 4 (from the beginning of extremity-calculus)

We consider a universe with a reflexive, antisymmetric, and transitive relation $\leq$. Furthermore, all suprema and infima are defined. (See Exercise 2.)

Let $\mathcal{B}$ be a predicate on our universe, and consider equation

(*) $\quad x : \; \mathcal{B}.x$

(This is our notation for an equation $\mathcal{B}.x$, in which x is the "unknown".)

a) Give a formal characterization of "equation (*) has a least (w.r.t. $\leq$) solution".

b) Prove that equation (*) has a least solution precisely whenever the infimum of all solutions solves (*)

Next, let f be an endofunction on our universe, and consider equation

(**) $\quad x : \; f.x \leq x$

c) Prove that for monotonic f, equation (**) has a least solution

d) [Theorem of Knaster & Tarski]
Prove that for monotonic f, equations

$$x : f.x \leq x \qquad \text{and}$$
$$x : f.x = x$$

have the same least solution

e) Let q be such that

$$f.q \leq q \qquad \text{and}$$
$$\langle \forall x : f.x = x : q \leq x \rangle \;.$$

Prove that, for monotonic f,
q is the least solution of $x : f.x \leq x$.

Exercise 5 (from relation calculus)

We extend the predicate calculus with a new binary infix operator, denoted ; ("semi"), of which we postulate that

- it is disjunctive (and hence monotonic, cf. Exercise 1e) in each of its arguments
- it has J as its two-sided identity element, i.e. $[x ; J \equiv x]$ and $[J ; x \equiv x]$.

Prove $\quad [x \Rightarrow J] \wedge [y \Rightarrow J] \Rightarrow [x ; y \equiv x \wedge y]$

(This has proven to be a difficult exercise for the uninitiated.)

Exercise 6 (from relation/regularity calculus)

We extend the predicate calculus with a new binary infix operator, denoted $;$ ("semi"), of which we postulate that

- it is associative
- it has J as its two-sided identity-element, i.e. $[x;J \equiv x]$ and $[J;x \equiv x]$
- it is universally disjunctive (and hence monotonic, cf. Exercise 1e) in each of its arguments.

We now consider, for arbitrary r, equation

(*) $\quad x : [J \vee x;r \Rightarrow x]$

Prove

a) Equation (*) has a strongest solution (i.e. a least solution with respect to $\Rightarrow$).

b) For any s, equation $x : [s;x \Rightarrow s]$ has a weakest solution.

c) For s the strongest solution of (*),
$[s;s \Rightarrow s]$ ("s is transitive")

d) Equations (*) and $x : [J \vee r;x \Rightarrow x]$ have the same strongest solution.