Predicate Logic vs Predicate Calculus :

It is up to you

We present two proofs of the theorem

$$(\langle \exists x : x \in V : P.x \rangle \Rightarrow \langle \forall y : y \in V : Q.y \rangle)$$
$$\equiv$$
$$\langle \forall x : x \in V : \langle \forall y : y \in V : P.x \Rightarrow Q.y \rangle \rangle .$$

Proof A, which I learned from a colleague, is carried out in Predicate Logic, and proof B, which I concocted myself, is rendered using Predicate Calculus.

Proof A (Verbatim)

"⇒"

1. Assume $\exists_{x \in V}(P(x)) \Rightarrow \forall_{y \in V}(Q(y))$

2. Assume $u \in V$

3. Assume $v \in V$

4. Assume $P(u)$

5. $\exists_{x \in V}(P(x))$ (IN* $\exists$ on 1 and 4)

6. $\forall_{y \in V}(Q(y))$ (EL $\Rightarrow$ on 1 and 5)

7. $Q(v)$ (EL $\forall$ on 6 and 3)

$P(u) \Rightarrow Q(v)$

$\forall_{v \in V}(P(u) \Rightarrow Q(v))$

$\forall_{u \in V} \forall_{v \in V}(P(u) \Rightarrow Q(v))$

"$\Leftarrow$"

1. Assume $\forall_{u\in V}\,\forall_{v\in V}\,(P(u)\Rightarrow Q(v))$
2. Assume $\exists_{x\in V}\,(P(x))$
3. Let $y\in V$
4. Let $x\in V$ with $P(x)$
5. $\forall_{v\in V}\,(P(x)\Rightarrow Q(v))$ (EL $\forall$ on 1 and 4)
6. $P(x)\Rightarrow Q(y)$ (EL $\forall$ on 5 and 3)
7. $Q(y)$ (EL $\Rightarrow$ on 4 and 6)
8. $Q(y)$ (EL* $\exists$ on 2, 4, and 7)

$\forall_{y\in V}\,(Q(y))$

$\exists_{x\in V}\,(P(x))\ \Rightarrow\ \forall_{y\in V}\,(Q(y))$

<u>End</u> of Proof A.

<u>Proof B</u> (Verbatim)

From Predicate Calculus we use – see Remark later on –

(0) For each Z and fresh dummy y, from whatever range,

$$(Z\Rightarrow\langle\forall y::Q.y\rangle)\ \equiv\ \langle\forall y::\ Z\Rightarrow Q.y\rangle$$

(1) For each Z and fresh dummy x, from whatever range,

$$(\langle\exists x::P.x\rangle\Rightarrow Z)\ \equiv\ \langle\forall x::P.x\Rightarrow Z\rangle\ .$$

Now the theorem follows from — V left anonymous —

$$\langle \exists x :: P.x \rangle \Rightarrow \langle \forall y :: Q.y \rangle$$
$$\equiv \quad \{ (1) \text{ with } Z := \langle \forall y :: Q.y \rangle \}$$
$$\langle \forall x :: P.x \Rightarrow \langle \forall y :: Q.y \rangle \rangle$$
$$\equiv \quad \{ (0) \text{ with } Z := P.x \}$$
$$\langle \forall x :: \langle \forall y :: P.x \Rightarrow Q.y \rangle \rangle \quad .$$

Remark Of course, the "fancy" (0) and (1) do the job. But how fancy are they? In lattice theory the infimum $\downarrow$ is defined by

$$(0') \qquad z \leq \langle \downarrow y :: q.y \rangle \quad \equiv \quad \langle \forall y :: z \leq q.y \rangle \quad ,$$

and (0) is just that in the lattice of predicates. The supremum $\uparrow$ is defined by

$$(1') \qquad \langle \uparrow x :: p.x \rangle \leq z \quad \equiv \quad \langle \forall x :: p.x \leq z \rangle \quad ,$$

which corresponds to (1). Are the definitions of $\downarrow$ and $\uparrow$ fancy things? I think they had better be in the foreground of our minds.
End of Remark.

End of Proof B

* * *

At my university, both "proof systems" are taught to CS-students — none of the two to Math-students ?!? — . I think there is no need to phrase or explain a point of view. It's all up to you.

WHJ Feijen
20 November 1996

PS I offer no apologies for having written this note.