Two "fusion-lemmata" on minimal solutions

(Nothing in this note is new.)

With $\mu\varphi$ the classical notation for the smallest (pre-)fix-point of a monotonic function, the first lemma can be formulated as

Rolling Rule

For monotonic f and g,

$$\mu(f \circ g) = f.(\mu(g \circ f)) .$$

Proof Let p and q be such that $p = \mu(f \circ g)$ and $q = \mu(g \circ f)$, then the theorem to be shown is

(*) $\quad p = f.q$.

The defining properties of p and q are

(0a) $\quad \langle \forall x : f.(g.x) \leq x : p \leq x \rangle$

(0b) $\quad f.(g.p) \leq p$

and

(1a) $\quad \langle \forall x : g.(f.x) \leq x : q \leq x \rangle$

(1b) $\quad g.(f.q) \leq q$

respectively.

We now prove (*) ping-pong-wise :

$$
\begin{array}{ll}
 & p \leq f.q \\
\Leftarrow & \quad \{ 0a \text{ with } x := f.q \} \\
 & f.(g.(f.q)) \leq f.q \\
\Leftarrow & \quad \{ f \text{ is monotonic} \} \\
 & g.(f.q) \leq q \\
\equiv & \quad \{ 1b \} \\
 & \text{true}
\end{array}
$$

$$
\begin{array}{ll}
 & f.q \leq p \\
\Leftarrow & \quad \{ 0b \text{ and transitivity of } \leq \} \\
 & f.q \leq f.(g.p) \\
\Leftarrow & \quad \{ f \text{ is monotonic} \} \\
 & q \leq g.p \\
\Leftarrow & \quad \{ 1a \text{ with } x := g.p \} \\
 & g.(f.(g.p)) \leq g.p \\
\Leftarrow & \quad \{ g \text{ is monotonic} \} \\
 & f.(g.p) \leq p \\
\equiv & \quad \{ 0b \} \\
 & \text{true} .
\end{array}
$$

<u>End</u> of Proof.

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For formulating the second lemma we have no classical notation at our disposal. It deals with a function $h.x.y$ that is monotonic in both arguments. We consider three equations, to wit

$y:$	$h.x.y \leq y$	with least solution $g.x$
$x:$	$g.x \leq x$	with least solution p
$x:$	$h.x.x \leq x$	with least solution q .

in terms of which the lemma can be formulated as

Diagonalization Rule

$$p = q$$

Proof The defining properties for g, p and q are

(0a) $\langle \forall y : h.x.y \le y : g.x \le y \rangle$ $(\forall x)$

(0b) $h.x.(g.x) \le g.x$ $(\forall x)$

and

(1a) $\langle \forall x : g.x \le x : p \le x \rangle$

(1b) $g.p \le p$ or $g.p = p$ (by Knaster/Tarski)

and

(2a) $\langle \forall x : h.x.x \le x : q \le x \rangle$

(2b) $h.q.q \le q$

respectively.

We now prove $p = q$ ping-pong-wise:

$$
\begin{array}{ll}
 & p \le q \\
\Leftarrow & \quad \{ \text{1a with } x := q \} \\
 & g.q \le q \\
\Leftarrow & \quad \{ \text{0a with } x, y := q, q \} \\
 & h.q.q \le q \\
\equiv & \quad \{ \text{2b} \} \\
 & \text{true}
\end{array}
$$

$$
\begin{array}{ll}
 & q \leq p \\
\Leftarrow & \quad \{ \text{2a with } x := p \} \\
 & h.p.p \leq p \\
\equiv & \quad \{ \text{1b, viz. } g.p = p \} \\
 & h.p.(g.p) \leq g.p \\
\equiv & \quad \{ \text{0b with } x := p \} \\
 & true \ .
\end{array}
$$

<u>End</u> of Proof

* * *

The reasons for recording this are
(i) that the lemmata and their proofs are quite nice in their own right, and
(ii) that we can beneficially use them to <u>derive</u> that

- regular expression $s ; *(t;s)$
  can be massaged into the equivalent
  $*(s;t);s$ — the "leap-frog rule",
- regular expression $*(s \vee t)$
  can be massaged into the equivalent
  $*t; *(s; *t)$ — "star-decomposition .

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