The Leap-Frog Rule and the Star-Decomposition, once more

In this note we record, for our own purposes, derivations of the Leap-Frog Rule and the Star-Decomposition from the regularity calculus, although this has been done, in different settings, in [0] and in [1]. We will use a notation that is a symbiosis of the notations in [0] and [1], and which exactly meets our manipulative requirements.

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For monotonic f, equations

$$x:\ f.x \leq x \qquad \text{and} \qquad x:\ f.x = x$$

have the same least solution (Knaster/Tarski), which we will denote

$$\langle \mu x : f.x \rangle \ .$$

In terms of this we can formulate the Rolling Rule and the Diagonal Rule as follows:

(RR) $\langle \mu x : f.(g.x) \rangle = f.\langle \mu x : g.(f.x) \rangle$,

(DR) $\langle \mu x : \langle \mu y : f.x.y \rangle \rangle = \langle \mu x : f.x.x \rangle$

* * *

As for the Leap-Frog Rule, we now observe

$$
\begin{array}{ll}
 & t;\ *(s;t) \\
\equiv & \quad \{\text{standard fixed-point}\} \\
 & \langle \mu x :\ t \vee x;s;t \rangle \\
\equiv & \quad \{\ ;\ \text{over}\ \vee\ \} \\
 & \langle \mu x :\ (J \vee x;s);t \rangle \\
\equiv & \quad \{\text{intro}\quad f = (;t) \\
 & \qquad\qquad\quad g = (J\vee)\circ(;s)\} \\
 & \langle \mu x :\ f.(g.x) \rangle \\
\equiv & \quad \{RR\} \\
 & f.\langle \mu x :\ g.(f.x) \rangle \\
\equiv & \quad \{\text{exit } f \text{ and } g\} \\
 & \langle \mu x :\ J \vee x;t;s \rangle;t \\
\equiv & \quad \{\text{standard fixed-point}\} \\
 & *(t;s);t
\end{array}
$$

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As for the Star-Decomposition, we observe

$$
\begin{array}{ll}
 & *(s \vee t) \\
\equiv & \quad \{\text{standard fixed-point}\} \\
 & \langle \mu x :\ J \vee (s \vee t);x \rangle \\
\equiv & \quad \{\ ;\ \text{over}\ \vee\} \\
 & \langle \mu x :\ J \vee s;x \vee t;x \rangle \\
\equiv & \quad \{DR\} \\
 & \langle \mu x :\ \langle \mu y :\ J \vee s;x \vee t;y \rangle \rangle \\
\equiv & \quad \{\text{standard fixed-point}\} \\
 & \langle \mu x :\ *t;(J \vee s;x) \rangle \\
\equiv & \quad \{\ ;\ \text{over}\ \vee\} \\
 & \langle \mu x :\ *t \vee *t;s;x \rangle \\
\equiv & \quad \{\text{standard fixed-point}\} \\
 & *(*t;s);*t\ .
\end{array}
$$

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[0] WF152 [1] Fixed-point calculus, RCB et.al.