A programming exercise communicated by Tom Verhoeff

Given a set S of N integers, $3 \le N$, we are asked to compute within the set of all $\binom{N}{3}$ triples the maximum value of

the absolute value of
the difference between
the triple's median and
the triple's average .

To put it more formally, for x, y, and z three (distinct) integers from S such that

$$x < y \ \wedge\ y < z \ ,$$

we have to maximize

$$\left| \, y - \frac{x+y+z}{3} \right| ,$$

or —equivalently—

$$| \, 2*y - x - z \, | ,$$

or — equivalently —

$$(2*y - x - z) \uparrow (x + z - 2*y) \ .$$

Because $\langle \uparrow i :: u.i \uparrow v.i \rangle = \langle \uparrow i :: u.i \rangle \uparrow \langle \uparrow i :: v.i \rangle$, we therefore have to compute the maximum of

(0a) $\langle \uparrow x,y,z : x < y < z : 2*y - x - z \rangle$ and

(0b) $\langle \uparrow x,y,z : x < y < z : x + z - 2*y \rangle$.

We focus on computing (0a) first.

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Because

$$(0a)$$

$$= \quad \{\text{nesting}\}$$

$$\langle \uparrow x :: \langle \uparrow y, z : x < y < z : 2 \times y - x - z \rangle \rangle$$

$$= \quad \{+ \text{ over } \uparrow\}$$

$$\langle \uparrow x :: -x \; + \; \langle \uparrow y, z : x < y < z : 2 \times y - z \rangle \rangle$$

we see that the maximum is obtained for $x := \downarrow S$ – the minimum of S – since the term is an anti-monotonic function of x. Hence

$$(0a) \quad = \quad -\downarrow S \; + \; \langle \uparrow y, z : \downarrow S < y < z : 2 \times y - z \rangle .$$

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As for the remaining quantified expression we observe that we can confine the computation to elements y and z that are "neighbours" in set S, i.e. have no elements in between them.

If our set S is given by the increasing array $f[0..N)$, we thus find

$$(0a) \quad = \quad -f.0 \; + \; \langle \uparrow i : 1 \le i < N-1 : 2 \times f.i \; - f.(i+1) \rangle$$

By the same token

$$(0b) \quad = \quad f.(N-1) + \langle \uparrow i : 1 \le i < N-1 : f.(i-1) - 2 \times f.i \rangle$$

The computation of $(0a) \uparrow (0b)$ has now become a standard programming exercise. With invariants

$$Pa: \qquad s = \langle \uparrow i : 1 \le i < n-1 : 2 \times f.i \; - f.(i+1) \rangle$$

Pb: $t = \langle \uparrow i : 1 \le i < n-1 : f.(i-1) - 2 \times f.i \rangle$

a solution is

$n := 2;\ s, t := -\infty, -\infty$
$;\ \{\text{inv. } Pa \wedge Pb\}$
do $n \ne N \rightarrow$
s, t
$:=$
$s \uparrow (2 \times f.(n-1) - f.n),\ \ t \uparrow (f.(n-2) - 2 \times f.(n-1))$
$;\ n := n+1$
od
$;\ max := (-f.0 + s) \uparrow (f.(N-1) + t)$

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