IMO 2007, Problem 4

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Problem Statement

The original problem statement reads:

Problem 4. In triangle $ABC$ the bisector of angle $BCA$ intersects the circumcircle again at $R$, the perpendicular bisector of $BC$ at $P$, and the perpendicular bisector of $AC$ at $Q$. The midpoint of $BC$ is $K$ and the midpoint of $AC$ is $L$. Prove that the triangles $RPK$ and $RQL$ have the same area.

Solution

Let’s start with a diagram (see Figure 1) presenting the givens.

Figure 1: Triangle $ABC$ (blue), and triangles $RPK$ and $RQL$ (red)
We calculate

\[
\text{area } RPK = \text{area } RQL
\]
\[
\equiv \{ KS \text{ is the altitude at } K; LT \text{ is the altitude at } L \}
\]
\[
\frac{1}{2}|RP| \cdot |KS| = \frac{1}{2}|RQ| \cdot |LT|
\]
\[
\equiv \{ \text{algebra, the lengths involved are nonzero} \}
\]
\[
|RP| : |RQ| = |LT| : |KS|
\]
\[
\equiv \{ \text{triangles } CPK \text{ and } CQL \text{ are similar, having two angles in common} \}
\]
\[
|RP| : |RQ| = |CQ| : |CP|
\]
\[
\equiv \{ P \text{ and } Q \text{ lie on } RC; \text{ rewrite } |CQ| \text{ and } |CP| \}
\]
\[
\]
\[
\equiv \{ \text{introduce names } a, b, c = |RP|, |RQ|, |CR| \}
\]
\[
a : b = c - b : c - a
\]
\[
\equiv \{ \text{algebra; the values involved are nonzero} \}
\]
\[
a(c - a) = b(c - b)
\]
\[
\equiv \{ \text{algebra} \}
\]
\[
a(c - a) - b(c - b) = 0
\]
\[
\equiv \{ \text{algebra} \}
\]
\[
(a - b)c - a^2 + b^2 = 0
\]
\[
\equiv \{ \text{algebra} \}
\]
\[
(a - b)(c - (a + b)) = 0
\]
\[
\equiv \{ \text{algebra} \}
\]
\[
a = b \lor c = a + b
\]
\[
\Leftarrow \{ a \text{ and } b \text{ are independent}, a, b, c = |RP|, |RQ|, |CR| \}
\]
\[
|CR| = |RP| + |RQ|
\]
\[
\equiv \{ |CR| = |RP| + |CP| \}
\]
\[
|RQ| = |CP|
\]

Therefore, it suffices to prove \(|RQ| = |CP|\). For that purpose, we introduce the line through the circumcenter \(M\) perpendicular to angle bisector \(CR\) (red in Figure 2). Now, reflect triangle \(ABC\) in this line to obtain triangle \(A'B'R\) (dashed blue in Figure 2).

Triangles \(ABC\) and \(A'B'R\) have the same circumcircle, since the reflection line passes through the circumcenter. Sides \(AC\) and \(B'R\) are parallel, because of equal angles with \(CR\). Similarly, sides \(BC\) and \(A'R\) are parallel. The perpendicular bisector of \(B'R\) passes through the circumcenter \(M\) and is
perpendicular to $AC$. Hence, the perpendicular bisectors of $AC$ and $B'R$ are the same. Consequently, $Q$ is the reflection of $P$ and, thus, $|RQ| = |CP|$. Q.E.D.

Figure 2: Triangle $ABC$ (blue), reflection line (red), reflected triangle $A'B'R$ (dashed blue)